Uniform Large Cardinal Characterizations and Ideals up to measurability (joint work with Philipp Lücke)

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10.07.2020

Ramsey cardinals

Victoria Gitman, Ian Sharpe and Philip Welch isolated the following from work of William Mitchell from the late 70'ies.

Theorem (Late 70ies, 2011)

κ is a Ramsey cardinal if and only if for every $x \subseteq \kappa$ there is a transitive weak κ-model M with $x \in M$ and with a (uniform) κ-amenable, countably complete and M-normal ultrafilter U on κ.

We require all our filters to be uniform: they only have elements of size κ.

- A *weak κ-model* M is a model of ZFC^- such that $|M| = \kappa$ and $\kappa + 1 \subseteq M$.
- An M-ultrafilter U is *M-normal* if it is closed under diagonal intersections in M, and *$<\kappa$-complete* if it is closed under $<\kappa$-intersections in M.
- U is *countably complete* if any countable intersection (in **V**) of elements of U is nonempty (equivalently, unbounded in κ).
- U is *κ-amenable* if whenever X is a set of size κ in M, then $X \cap U \in M$.

Varying the parameters

What happens if we vary the requirements on M and on U? For example:

Theorem

κ is weakly compact iff for all $x \subseteq \kappa$ there is a transitive weak κ-model M with $x \in M$ and a κ-amenable $<\kappa$-complete M-ultrafilter U on κ.

Remember that the following are equivalent to κ being weakly compact:

- κ has the *filter property*: whenever $\mathcal{A}$ is a κ-sized collection of subsets of κ, there is a $<\kappa$-complete ultrafilter U that measures all sets in $\mathcal{A}$
- κ has the *filter extension property*: if U is a $<\kappa$-complete ultrafilter measuring at most κ-many subsets of κ, and $\mathcal{A}$ is a κ-sized collection of subsets of κ, then there is a $<\kappa$-complete ultrafilter $V \supseteq U$ that measures $\mathcal{A}$

Letting $x \subseteq \kappa$ code $\mathcal{A}$ in the above theorem, the statement in the theorem clearly yields a $<\kappa$-complete ultrafilter that measures $\mathcal{A}$, i.e. it implies the weak compactness of κ.

Proof continued

For the other direction, assume that κ is weakly compact and that $x \subseteq \kappa$. We need to find a weak κ-model M with $x \in M$ and a κ-amenable $<\kappa$-complete M-ultrafilter U on κ. We construct ω-sequences $\langle M_n \mid n < \omega \rangle$ of weak κ-models $M_n \prec H(\kappa^+)$ and $\langle U_n \mid n < \omega \rangle$ of $<\kappa$-complete M_n-ultrafilters on κ. Let M_0 be such that $x \in M_0$ and let U_0 be the cobounded filter on κ. Assume that M_n and U_n are constructed, let M_{n+1} be such that $M_n, U_n \in M_{n+1}$, and using the filter extension property, let $U_{n+1} \supseteq U_n$ be a $<\kappa$-complete M_{n+1}-ultrafilter. Let $M = \bigcup_{n<\omega} M_n$ and $U = \bigcup_{n<\omega} U_n$. Then, U is a $<\kappa$-complete ultrafilter for the weak κ-model $M \prec H(\kappa^+)$. If $\vec{x} \in M$ is a sequence of subsets of κ in M, then it is in some M_n, hence each of its sequents is measured by $U_n \subseteq U$. Thus, by our choice of M_{n+1}, U restricted to $\vec{x}$ is an element of $M_{n+1} \subseteq M$, i.e. U is κ-amenable for M.

More variations

Theorem (Reminder)

κ is a Ramsey cardinal if and only if for every $x \subseteq \kappa$ there is a transitive weak κ-model M with $x \in M$ and with a κ-amenable, countably complete and M-normal ultrafilter U on κ.

- Instead of the countable completeness of U, only require the ultrapower of M by U to be well-founded.
- Do not require well-foundedness of the ultrapower.

Or require U to be ...

- *stationary-complete*: Every countable intersection from U (in $\mathbf{V}$) is stationary in κ.
- *genuine*: Every diagonal intersection of elements of U is unbounded in κ.
- *normal*: Every diagonal intersection of U is stationary in κ.

We may also require that $M \prec H(\theta)$ for sufficiently large regular θ instead of transitivity of M in any of the above.

A table of results and definitions

U is κ-amenable and...	M is transitive	$M \prec H(\theta)$
$<\kappa$-complete for M	weakly compact	weakly compact
M-normal	$\mathbf{T}^\kappa_\omega$-Ramsey	completely ineffable
... and well-founded	weakly Ramsey	ω-Ramsey
... and countably complete	Ramsey	$\prec$-Ramsey
... and stationary-complete	ineffably Ramsey	Δ-Ramsey
genuine	∞^κ_ω-Ramsey	Δ-Ramsey
normal	Δ^κ_ω-Ramsey	Δ-Ramsey

An example on how to read the above table:

κ is completely ineffable iff for every sufficiently large regular θ and every $x \in H(\theta)$ there is a weak κ-model $M \prec H(\theta)$ with $x \in M$ and with a κ-amenable, M-normal ultrafilter U on κ.

This particular result is actually a consequence of results by myself and Philipp Schlicht, and by Dan Nielsen and Philip Welch.

Completely ineffable and completely Ramsey cardinals

Definition

$\mathcal{S} \subseteq \mathcal{P}(\kappa)$ is a *stationary class* if $\mathcal{S} \neq \emptyset$ is a collection of stationary subsets of κ.

Definition

A cardinal κ is *completely ineffable* if there is a stationary class $\mathcal{S} \subseteq \mathcal{P}(\kappa)$ such that whenever $A \in \mathcal{S}$ and $f\colon [A]^2 \to 2$, then there is $H \subseteq A$ in $\mathcal{S}$ that is homogeneous for f.

Definition

A cardinal κ is *completely Ramsey* if there is a stationary class $\mathcal{S} \subseteq \mathcal{P}(\kappa)$ such that whenever $A \in \mathcal{S}$ and $f\colon [A]^{<\omega} \to 2$, then there is $H \subseteq A$ in $\mathcal{S}$ that is homogeneous for f.

Question: How do completely Ramsey cardinals fit with this table?

Uniform large cardinal ideals

These large cardinal characterizations also allow for highly uniform definitions of corresponding *large cardinal ideals*. Let φ denote a large cardinal property that is characterized through the existence of certain models M (either transitive weak κ-models, or weak κ-models $M \prec H(\theta)$) with M-ultrafilters U having a certain property φ^*. We define I_φ and $I_{\prec\varphi}$ as follows:

- $A \in I_\varphi$ if there is $x \subseteq \kappa$ such that for all transitive weak κ-models M with $x \in M$ and every M-ultrafilter U with Property φ^*, $A \notin U$.
- $A \in I_{\prec\varphi}$ if for all sufficiently large regular θ there is $x \in H(\theta)$ such that for all weak κ-models $M \prec H(\theta)$ with $x \in M$ and every M-ultrafilter U with Property φ^*, we have $A \notin U$.

Given that $\varphi(\kappa)$ holds, I_φ and $I_{\prec\varphi}$ are easily seen to be proper ideals on κ. If φ^* implies the M-normality of U, then they are normal ideals on κ.

Established large cardinal ideals

In all cases of large cardinals for which corresponding large cardinal ideals had already been defined, these coincide with our definitions: Ramsey, completely ineffable, ineffably Ramsey. Also - using a different characterization than the one I mentioned - weakly compact, plus also weakly ineffable and ineffable (which I haven't mentioned yet at all).

Often, these ideals correspond to natural and well-known set-theoretic objects. For example, let κ be completely ineffable. An adaption of the proofs mentioned above yields the following.

Theorem

The completely ineffable ideal is the complement of the $\supseteq$-maximal stationary class witnessing the complete ineffability of κ.

Hierarchy results

We can show in most cases that proper containment of large cardinal ideals corresponds to their ordering with respect to direct implication. For example: Weakly compact ideal $\subsetneq$ Ineffable Ideal $\subsetneq$ Completely Ineffable ideal $\subsetneq$ weakly Ramsey ideal $\subsetneq$ Ramsey ideal $\subsetneq$ $\prec$-Ramsey ideal $\subsetneq$ *measurable ideal*.

Moreover, we can also show that the ordering of large cardinals with respect to consistency strength reflects to a property of their corresponding ideals in many cases - given large cardinal notions A consistency-wise weaker than B, $B(\kappa)$ implies that the set $\{\lambda < \kappa \mid \neg A(\lambda)\}$ is in the B-ideal on κ.

For example, Ramsey cardinals are consistency-wise stronger than completely ineffable cardinals, but need not even be ineffable themselves. In this case, it follows by a result of Gitman that if κ is a Ramsey cardinal, then the non-completely ineffables below κ are in the Ramsey ideal on κ.

The measurable ideal

The *measurable ideal* I^κ_{ms} on a measurable cardinal κ is defined as well by the uniform framework from our paper, and turns out to be the complement of the union of all normal ultrafilters on κ. This ideal is not very interesting in small inner models (for example in $L[U]$). Moreover:

Theorem

If any set of pairwise incomparable conditions in the Mitchell ordering at κ has size at most κ, then the partial order $\mathcal{P}(\kappa)/I^\kappa_{ms}$ is atomic.

However, it is consistently non-trivial – adapting classical arguments from Kunen and Paris yields the following:

Theorem

Every model with a measurable cardinal κ has a forcing extension in which $\mathcal{P}(\kappa)/I^\kappa_{ms}$ is atomless.

Atomicity for smaller large cardinals

Theorem

If I is a normal ideal on a regular and uncountable cardinal κ such that the partial order $\mathcal{P}(\kappa)/I$ is atomic, then κ is measurable and $I^{\kappa}_{ms} \subseteq I$.

Thus, for many large cardinal notions below measurability, we can infer that their induced ideals are never atomic: Assume that κ were such a large cardinal. If κ is not measurable, then we are done by the above theorem. If κ is measurable, then for many large cardinal notions, our results show that their induced ideals are properly contained in the measurable ideal. Therefore, by the above theorem, we are again done.

Normally Ramsey cardinals

Definition

An uncountable cardinal κ is S-Ramsey / ∞-Ramsey / Δ-Ramsey if for every regular $\theta > \kappa$, every $x \in H(\theta)$ is contained in a weak κ-model $M \prec H(\theta)$ with a κ-amenable, M-normal ultrafilter U on κ that is stationary-complete / genuine / normal.

Generalizing results from Holy and Schlicht shows the following.

Theorem

κ is S-Ramsey / ∞-Ramsey / Δ-Ramsey if for all regular $\theta > \kappa$, Player I does not have a winning strategy in the game of length ω in which Player I plays a $\subset$-increasing sequence of κ-models $M_i \prec H(\theta)$ with union M, and Player II responds with a $\subseteq$-increasing sequence of M_i-ultrafilters U_i with union U. Player I also has to ensure that M_i and U_i are both elements of M_{i+1} for every $i \in \omega$. Player II wins if U is an M-normal filter that is stationary-complete / genuine / normal.

… are equivalent to some seemingly weaker Ramsey-like cardinals

Lemma

S-Ramsey $\equiv \infty$-Ramsey $\equiv \Delta$-Ramsey.

Proof: Assume that κ is S-Ramsey, that $\theta > \kappa$ is regular, and let $x \in H(\theta)$. Let $M_0 \prec H(\theta)$ with $x \in M_0$ be a weak κ-model. Consider a run of the game for S-Ramseyness, in which Player I starts by playing M_0, and which Player II wins – with resulting model $M = \bigcup_{i<\omega} M_i$ and M-ultrafilter $U = \bigcup_{i<\omega} U_i$. This means that $M \prec H(\theta)$ is a weak κ-model with $x \in M$, and U is κ-amenable, M-normal and stationary-complete. But $\Delta U \supseteq \bigcap_{i<\omega} \Delta U_i$ (modulo a non-stationary set). Since each $\Delta U_i \in U$, it follows that ΔU is stationary, for it is stationary-complete. But this means that U is normal, and hence κ is Δ-Ramsey. □

Questions

Question

We can only verify our structural results on a case by case basis. However, do they hold below measurability in general? Or, are there any counterexamples?

Question

Can similar things be done for large cardinals above measurability?